cremental indentation step. This fact has been stated in the first paragraph of the introduction of Ref. 2 with a typical stress-strain curve for material with strain hardening and the implications that arise from the relative amount of incremental loading and the slope of the stress-strain curve, therefore, the degree of positive definiteness of the stiffness matrix obtained has been explained with reference to Fig. 2; and one method of diagnosis for ill behavior of the stiffness matrix has been explained with reference to Fig. 3 on page 1827 of Ref. 2.

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A Further Note on Laminar Incipient Separation

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Nomenclature

H = boundary-layer form factor
K = pressure gradient parameter
M = Mach number
S_w = total enthalpy function
T = temperature
τ = shear stress
θ_i = incipient compression surface deflection angle
$\bar{\chi}$ = viscous interaction parameter

Subscripts

o = flat plate, i.e., $K_o = 0$
s = separation, i.e., $\tau_w = 0$
tr = transformed
w = wall

Reference 1 presented the effect of wall temperature on the incipient deflection angle, θ_i, i.e., that angle which the laminar boundary layer can negotiate without separating. The wall temperature was shown to enter into the valuation of λ in the equation

$$M\theta_i = \lambda\bar{\chi}^{1/2} \tag{1}$$

where

$$\lambda = -2.45K_s\,(-\Delta H_{tr}/\Delta K)^{1/2} \tag{2}$$

Although λ may be evaluated accurately through Eq. (2) by making use of the similar solutions of Ref. 2, its physical significance was not apparent.

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Recalling that

$$\Delta H_{tr} \equiv H_{tr_s} - H_{tr_o}$$
$$\Delta K \equiv K_0 - K_s = -K_s,\ K_o = 0$$

and noting that, very accurately, independent of the wall-to-stagnation temperature ratio,

$$K_sH_{tr_s} \cong -0.275, \quad -1 \le S_w \le 1$$

one obtains, upon substitution into Eq. (2),

$$\lambda = 1.288(1 - H_{tr_o}/H_{tr_s})^{1/2} \tag{3}$$

Therefore, λ is a function of the ratio of the transformed form factor at the beginning of the interaction and at the point of incipient separation. Finally, it is noted that the ratio H_{tr_o}/H_{tr_s} increases nonlinearly with S_w only through H_{tr_s} since $H_{tr_o} = 2.591\ (S_w + 1)$, H_{tr_s} being inversely proportional to the pressure gradient at separation.

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Comment on "Exact Solution of Certain Problems by Finite Element Method"

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In his recent Note, Tong[1] proved that when the exact solution to the homogeneous Euler Equations of a positive definite functional[2] with one independent variable is known, and it is used as interpolation functions in the variational formulation of the finite element equations, the generalized displacements which are the solution to these equations constitute the exact solution to the problem at the nodal points regardless of the number or size of the elements used in the discretization. This property is often used in many one-dimensional problems and also in two-dimensions when a separation of variables is applicable. Typical problems of this kind are those concerning continuous beams, frame structures, some plate problems, and axisymmetric shells of revolution.

It is of interest to note, however, that an alternate derivation of the finite element equations has been used by engineers before the advent of the finite element method as such. If the generalized displacements are defined as the displacements at the element ends, the Euler Equations of the appropriate functional are equilibrium equations at the same points. In Tong's Note these are given in matrix form in Eq. (8). Obviously the entries of the stiffness matrix **K** are the end forces corresponding to unit values of the end displacements, and they can be obtained from the general solution to the homogeneous equations. Thus, when these end forces are obtained from the exact general solution, they are exact too.

Then, it remains to prove that the generalized forces in Eq. (8), **Q**, are also exact. The generalized forces are defined

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